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# Integration-by-Parts Formulas for Boundary-Element Methods

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**The numerical solution of problems in potential flow or linear elasticity via boundary-element methods (panel methods) usually confronts the investigator with strongly singular, "hypersingular" integrals. The well-known line vortex integration-by-parts procedure of potential flow uses Stokes' theorem to isolate the most singular part of the hypersingular integral as a discardable line integral, the line vortex term. This procedure is extended to identify a more weakly singular line integral, the "edge jet" term. The full integration-by-parts procedure is extended to the theory of linear elasticity, identifying "edge dislocation" and "stress-jump" line integrals. In all cases, elimination of the singular line integrals is seen to depend on the imposition of matching (or continuity) conditions. Procedures for the nonredundant imposition of such matching conditions are outlined. The paper concludes with a description of how the integration-by-parts procedure may be generally applied in other contexts.**

## Nomenclature

$a_\infty$ = freestream speed of sound
$B$ = region in which $u_i$ satisfies equations of linear elasticity
$c_\alpha$, $c_1$, $c_2$ = wave speeds in an elastic medium, $c_1$ for pressure waves and $c_2$ for shear
$G_{kj}$ = Somilgian tensor in elasticity
$g$ = distance from a field point $x'$ to a source point $x \in \partial Q$
$L_{ik}$ = operator $(\lambda + \mu)\partial_i\partial_k + \mu\delta_{ik}\nabla^2$ of linear elasticity
$M$ = Mach number, $V/a_\infty$
$n$, $n_i$ = unit normal vector to $S$ and its elements, $Q$
$\tilde{n}$, $\tilde{n}_i$ = conormal vector formed from $n$ and its components
$Q$ = panel (element) that forms a part of $S$
$Q_k$ = $k$th panel participating in an abutment
$\partial Q$ = boundary of $Q$
$R$ = compressible distance between field point $x'$ and source point $x$
$s_k$ = orientation sign of panel $Q_k$ relative to an abutment
$S$, $S_+$, $S_-$ = singularity surface, its upper surface, and its lower surface
$dS_x$ = element of area on the singularity surface $S$
$t_{ijk}$ = stress operator of linear elasticity
$T$, $T_i$ = surface traction vector and its components
$T_i^{(t)}$ = that part of $T_i$ computable from the tangential derivative of $u_k$ on $S$
$T_i^{(n)}$ = $T_i - T_i^{(t)}$
$u$, $u_i$ = elastic displacement and its components
$u_i^Q$ = contribution of panel $Q$ to the representation formula for $u_i$
$u_i$ = jump across $S$ of the conormal derivative of $\partial_i\phi$
$V$ = freestream velocity in the $x_1$ direction
$w$, $w_i$ = velocity jump across $S$ and its components
$x$, $x_i$ = source point and its components
$x'$, $x_i'$ = field point and its components
$dx$, $dx_i$ = element of arclength along $\partial Q$
$\delta(x - x')$ = usual Dirac delta function of a vector argument
$\delta_{ij}$ = Kronecker delta, 1 if $i = j$, 0 otherwise
$\epsilon_{ijk}$ = usual permutation symbol on three indices
$\iota$ = imaginary unit
$\lambda$, $\mu$ = Lamé parameters of linear elasticity
$\mu$ = doublet strength of potential flow, $[\phi]$
$\rho$ = density of the elastic solid
$\sigma$ = source strength of potential flow, $[\partial\phi/\partial\tilde{n}]$
$\tau$, $\tau_{ij}$ = stress tensor and its components
$\phi$ = potential function of potential flow
$\phi^Q$ = contribution of panel $Q$ to the representation formula for $\phi$
$\psi$ = fundamental singularity function of potential flow
$\psi_\alpha$ = fundamental singularity functions at wave speeds $c_1$ and $c_2$
$\omega$ = angular frequency of time harmonic motion
$\Omega$ = Helmholtz parameter of the Prandtl-Glauert-Helmholtz equation
$\partial_i$ = $\partial/\partial x_i$ partial derivative operator with respect to the source point $x_i$
$\partial_i'$ = $\partial/\partial x_i'$ partial derivative operator with respect to the field point $x_i'$
$\nabla$ = gradient operator having components $\partial_i = \partial/\partial x_i$
$\tilde{}$ = modifying symbol; for any vector $a$, $\tilde{a} = [(1 - M^2)a_1, a_2, a_3]$
[ ] = symbols denoting the jump of the enclosed quantity from $S_-$ to $S_+$

## I. Introduction

AT the International Symposium on Boundary Element Methods in 1989, it became apparent that some techniques well known to panel method workers were not so widely disseminated as might have been expected. This paper is intended to correct that problem by summarizing some results for potential flow and extending those results to linear elasticity. The paper, consisting of four sections, is summarized as follows.

In Sec. II, the integral representation formula and corresponding integration-by-parts procedures associated with the equation governing linearized, time harmonic, unsteady potential flow[1,2] will be stated and discussed. The integration-by-parts formula expresses the gradient of the potential function $\phi$ as a sum of surface integrals and line integrals. One of the line-integral terms is the well-known line vortex term. A second "edge jet" term seems not to be so well known.

In Sec. III, the time-harmonic Navier equations of linear elasticity will be treated similarly. The integral representation formula (the Betti identity) for the displacement field $u_k$ will be stated and then the integration-by-parts formula for the displacement gradient will be given. The line-integral terms analogous to the line vortex and edge-jet terms will be clearly identified. This integration procedure appears to be new, and should provide considerable help in dealing with the "hypersingular" integrals that cause some difficulty in the implementation of a collocation procedure.

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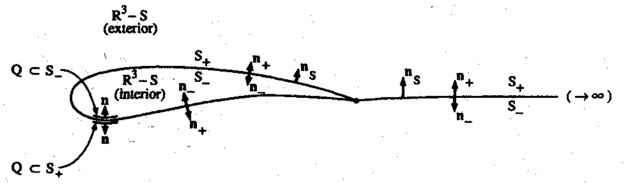


**Fig. 1 Concepts of the potential representation formula (3).**

In Sec. IV, the matching conditions used to eliminate the singular line integrals are discussed. The integration-by-parts formulas given in the previous two sections provide integral representations for the potential gradient in potential flow, and for the displacement gradient in linear elasticity, the formulas in each case containing both surface and line integral terms. Provided that appropriate matching conditions are imposed on the surface singularity functions, edge contributions made by adjacent panels will be equal and opposite and will cancel. In this way, the more strongly singular line integrals can be discarded. The nature of the matching conditions and how they may be correctly imposed in a collocation procedure that decomposes the singularity surface into networks of panels will then be summarized. In particular, the consistent imposition of matching conditions at an abutment intersection, where several networks come together at a point, will be shown to depend upon the construction of a directed graph that captures the essential topology of the abutment intersection. The matching conditions will then be seen to be equivalent to Kirchhoff current laws for the nodes of the graph, and a consistent set of matching condition assignments will be obtained by prescribing an orderly defoliation of a spanning tree for the graph.

Section V closes the paper with a short discussion of what I consider to be the key insights that aid in the derivation of useful integration-by-parts formulas.

## II. Potential Flow Analysis

In this section, we state the integral representation result for the time-harmonic unsteady potential flow equation:

$$[(1-M^2)\partial_1^2 + \partial_2^2 + \partial_3^2 + \Omega^2(1-M^2)]\phi = 0 \tag{1}$$

apply the gradient operator to obtain a representation for $\partial_i\phi$ and then apply Stokes' theorem in a rather special way to decompose the representation of $\partial_i\phi$ into the desired line and surface integrals.

We begin our discussion by pointing out that the potential function $\phi$ of Eq. (1) is not exactly the velocity potential for the unsteady velocity potential. That function, denoted $\Phi$, satisfies the wave equation in a moving frame given by Garrick,[3]

$$[(\partial_1^2 + \partial_2^2 + \partial_3^2) - (1/a_\infty^2)(\iota\omega + V\partial_1)^2]\Phi = 0 \tag{2}$$

where here and elsewhere we use the shorthand $\partial_i$ to denote the operator $\partial/\partial x_i$. The auxiliary potential $\phi$ is connected to $\Phi$ by the relation

$$\phi = e^{-\iota\Omega M x_1}\Phi$$

where the Helmholtz parameter $\Omega$ is defined in terms of the oscillatory frequency $\omega$:

$$\Omega = \omega M/[V(1-M^2)]$$

The integral representation formula for $\phi$ is equivalent to the classical Helmholtz theorem as given in Bergmann and Schiffer[4] when $M<1$. For $M>1$, a much more careful analysis is required, Morino[1] providing an excellent discussion. The results of those analyses may be stated as follows. Any function $\phi$ satisfying Eq. (1) in a region of space $(R^3-S)$ may be expressed in terms of the behavior of $\phi$ in the neighborhood of the singularity surface $S$ by the basic representation formula,

$$\phi(x') = \iint_S \sigma(x)\psi(x',x)\,\mathrm{d}S_x - \iint_S \mu(x)\frac{\partial\psi}{\partial\tilde{n}}\,\mathrm{d}S_x \tag{3}$$

where (see Fig. 1 for graphical descriptions) $S$ is the singularity surface for the function $\phi$; $x'$ is the field point at which $\phi$ is evaluated; $x$ is the source point, which is restricted to $S$; and $n$ is the normal to $S$, pointing into $(R^3-S)$ from the upper surface of $S_+$. Further, we have the definitions

$$\tilde{n} = [(1-M^2)n_1, n_2, n_3] \tag{4a}$$

for the modified normal (conormal) to $S$,

$$\sigma = (\partial\phi/\partial\tilde{n})_+ - (\partial\phi/\partial\tilde{n})_- \tag{4b}$$

for the jump in $\tilde{n}\cdot\nabla\phi$ across $S$ (source strength),

$$\mu = \phi_+ - \phi_- \tag{4c}$$

for the jump in $\phi$ across $S$ (doublet strength), and where

$$\psi = \psi(x'-x) \tag{4d}$$

for the kernel function appropriate to the governing partial differential equation.

The kernel function $\psi$ is a function of the relative position vector $(x'-x)$ and is given by different formulas for the subsonic and supersonic cases. For $M<1$, $\psi$ is easily derived from the classical Helmholtz kernel, and is given by

$$\psi = \psi(R) = -e^{-\iota\Omega R}/(4\pi R) \tag{5}$$

where $R$ is defined by

$$R^2 = (x_1'-x_1)^2 + (1-M^2)[(x_2'-x_2)^2 + (x_3'-x_3)^2] \tag{6}$$

For $M>1$, $\psi$ must be set equal to zero for $x$ outside of the domain of dependence of $x'$. The formula reads

$$\psi = \psi(R) = \begin{cases} -\cos(\Omega R)/(2\pi R) & x_1 < x_1' - \sqrt{M^2-1}\,\gamma \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

with $R^2$ again given by Eq. (6) and $\gamma^2 = (x_2'-x_2)^2 + (x_3'-x_3)^2$.

In the supersonic case, all integrals must be interpreted in the sense of finite-part integration. In point of fact, this requirement will not affect our presentation, the rules for interchanging differentiation with finite-part integration being exactly what one would expect. In particular, the distributional version of Stokes' theorem that must be used with finite-part integration is formally unchanged from the usual version.

Before proceeding with our derivations, we must define some notation and make some important observations. As before, we use $\partial_i$ to denote $\partial/\partial x_i$, the gradient operator with respect to the source point variables. Differentiation with respect to the field point variables, $\partial/\partial x_i'$, is denoted by $\partial_i'$. Given a vector $a$ having components $a_i$, a modified vector $\tilde{a}$ having components $\tilde{a}_i$ is obtained by setting $\tilde{a} = [(1-M^2)a_1, a_2, a_3]$, that is, by multiplying the first component by $(1-M^2)$. Note that the modified normal $\tilde{n}$ was formed in exactly this manner. Using the modified gradient operator $\tilde{\partial}_i$, the governing differential equation (1) can be re-expressed as

$$[\partial_i\tilde{\partial}_i + \Omega^2(1-M^2)]\phi = 0 \tag{8}$$

Because the kernel $\psi$ is a function of $(x'-x)$, we have the antisymmetry property,

$$\partial_i'\psi = -\partial_i\psi \tag{9}$$

Also, because the kernel function $\psi$ is a fundamental singularity for the differential equation (8), it satisfies the relations

$$[\partial_i \tilde{\partial}_i + \Omega^2(1-M^2)]\psi = [\partial_i' \tilde{\partial}_i' + \Omega^2(1-M^2)]\psi = \delta(x-x') \quad (10)$$

We begin our derivation by recasting Eq. (3) in a form that will better motivate our derivation. Denoting the boundary of $(R^3 - S)$ by $S_+ \cup S_-$, the union of the upper and lower surface of $S$, the basic representation formula (3) may be recast in the form

$$\phi(x') = \iint_{S_+ \cup S_-} \left[\frac{\partial \phi}{\partial \tilde{n}} \psi(x',x) - \phi(x) \frac{\partial \psi}{\partial \tilde{n}}\right] dS_x \quad (11)$$

where the evaluation of $\phi$ and $\partial\phi/\partial\tilde{n}$ on $S_+ (S_-)$ is performed by evaluating the limit of these as $\epsilon \to 0$ at points $x + \epsilon n (x - \epsilon n)$. Also, the meaning of $n$ has changed slightly, now being interpreted as the normal that points into $(R^3 - S)$ from a point $x$ on the surface $S_+$ or $S_-$. Now in most conventional panel method implementations, the integral over the surface $S_+ \cup S_-$ is calculated as the sum of integrals over surface elements $Q \subset S_+$ (or $Q \subset S_-$), the panels of the configuration. Thus, we are led to consider a panel's contribution to the representation formula (11), which we denote by $\phi^Q(x')$:

$$\phi^Q(x') = \iint_Q n_j [\psi\, \tilde{\partial}_j \phi - \phi\, \tilde{\partial}_j \psi]\, dS_x \quad (12)$$

Notice the prominence of the expression $[\psi\tilde{\partial}_j\phi - \phi\tilde{\partial}_j\psi]$ in this equation. This expression, which is of fundamental importance in the derivation of the basic representation formula, is also of fundamental importance in the derivation of the integration-by-parts formula. Applying the operator $\partial_k'$ to Eq. (12), interchanging the order of integration and differentiation and employing Eq. (9), we obtain

$$\partial_k' \phi^Q = \frac{\partial \phi^Q(x')}{\partial x_k'} = -\iint_Q n_j [\tilde{\partial}_j \phi\, \partial_k \psi - \phi\, \partial_k \tilde{\partial}_j \psi]\, dS_x \quad (13)$$

We remark that the second term appearing in the integral on the right causes significant difficulty in that the function $\partial_k \tilde{\partial}_j \psi$ behaves like $1/\|x - x'\|^3$ as $x' \to x$. The resulting surface integrals are not properly convergent in the limit, leading to what have been called hypersingular integrals. To deal with this problem, we use Stokes' theorem in the form,

$$\int_{\partial Q} dx_i \epsilon_{ijk} f = \iint_Q (n_j \partial_k - n_k \partial_j) f\, dS_x \quad (14)$$

where $\epsilon_{ijk}$ is the usual presentation symbol. [Remark: This particularly useful form of Stokes' theorem is obtained from the more usual statement, $\int_{\partial Q} dx_i F_i = \iint_Q \epsilon_{pqi} n_p \partial_q F_i\, dS_x$, by setting $F_i = \epsilon_{ijk} f$ and using the identity $\epsilon_{ipq}\epsilon_{ijk} = \delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj}$.] Setting $f = \psi\tilde{\partial}_j\phi - \phi\tilde{\partial}_j\psi$ in Eq. (14), we obtain

$$\begin{aligned}
&\int_{\partial Q} dx_i \epsilon_{ijk} [\psi\tilde{\partial}_j\phi - \phi\tilde{\partial}_j\psi]\, dS_x \\
&= \iint_Q (n_j\partial_k - n_k\partial_j)[\psi\tilde{\partial}_j\phi - \phi\tilde{\partial}_j\psi]\, dS_x \\
&= \iint_Q [(n_j\partial_k - n_k\partial_j)\psi]\tilde{\partial}_j\phi\, dS_x + \iint_Q \psi(n_j\partial_k - n_k\partial_j)\tilde{\partial}_j\phi\, dS_x \\
&- \iint_Q [(n_j\partial_k - n_k\partial_j)\phi]\tilde{\partial}_j\psi\, dS_x - \iint_Q \phi(n_j\partial_k - n_k\partial_j)\tilde{\partial}_j\psi\, dS_x
\end{aligned} \quad (15)$$

Now the second and sixth terms on the right cancel one another by virtue of the identity $(\partial_j\psi)(\tilde{\partial}_j\phi) = (\tilde{\partial}_j\psi)(\partial_j\phi)$. Rearranging what remains, we get

$$\begin{aligned}
\int_{\partial Q} dx_i \epsilon_{ijk} [\psi\tilde{\partial}_j\phi - \phi\tilde{\partial}_j\psi] &= \iint_Q n_j [\tilde{\partial}_j\phi\partial_k\psi - \phi\partial_k\tilde{\partial}_j\psi]\, dS_x \\
&+ \iint_Q n_k [\phi\partial_j\tilde{\partial}_j\psi - \psi\partial_j\tilde{\partial}_j\phi]\, dS_x \\
&+ \iint_Q [\psi(n_j\tilde{\partial}_j\partial_k\phi) - \partial_k\phi(n_j\tilde{\partial}_j\psi)]\, dS_x
\end{aligned} \quad (16)$$

From Eq. (13), we recognize the first line on the right as $-\partial_k'^Q$. The second line vanishes by virtue of Eqs. (8) and (10) (as long as $x' \notin Q$). The third line is exactly what we would get for a panel's contribution to $\partial_k'\phi$ by applying the representation formula (11) to $\partial_k\phi$, a function that also satisfies Eq. (8). We obtain thus,

$$\begin{aligned}
\partial_k'\phi^Q &= \iint_Q [\psi(n_j\tilde{\partial}_j\partial_k\phi - \partial_k\phi(n_j\tilde{\partial}_j\psi)]\, dS_x \\
&- \int_{\partial Q} dx_i \epsilon_{ijk} [\psi\tilde{\partial}_j\phi - \phi\tilde{\partial}_j\psi]
\end{aligned} \quad (17)$$

A slightly different form of this relation can be obtained if we do not delete the second line of Eq. (16). We obtain in this way [using the result $\partial_j\tilde{\partial}_j\psi = -(1-M^2)\Omega^2\psi$],

$$\begin{aligned}
\partial_k'\phi^Q &= \iint_Q \{\psi[(n_j\partial_k - n_k\partial_j)\tilde{\partial}_j\phi - (1-M^2)\Omega^2 n_k\phi] \\
&- (\partial_k\phi)\,(n_j\tilde{\partial}_j\psi)\}\, dS_x - \int_{\partial Q} dx_i \epsilon_{ijk} [\psi\tilde{\partial}_j\phi - \phi\tilde{\partial}_j\psi]
\end{aligned} \quad (18)$$

This form of our result is useful when we convert back to a source/doublet representation by combining the contribution made by $Q$ considered as being part of $S_+$ with a corresponding contribution made by $Q \subset S_-$. When this is done, the right side of Eq. (18) transforms according to the rules

$$\phi \mapsto [\phi] = \mu, \qquad \partial_j\phi \mapsto [\partial_j\phi] = w_j \quad (19)$$

The vector $w = \{w_j\}$, the jump in $\nabla\phi$, can be expressed in terms of $\sigma$ and $\mu$ as follows. Consider the vector identity

$$(n \cdot \tilde{n})\nabla\phi = (n \times \nabla\phi) \times \tilde{n} + n(\tilde{n} \cdot \nabla\phi)$$

Evaluating this above and below the surface $S$ and computing the difference yields

$$\begin{aligned}
(n \cdot \tilde{n})w &= (n \cdot \tilde{n})[\nabla\phi] = (n \times [\nabla\phi]) \times \tilde{n} + n\left[\frac{\partial\phi}{\partial\tilde{n}}\right] \\
&= (n \times \nabla\mu) \times \tilde{n} + n\sigma
\end{aligned} \quad (20)$$

where we have used the definition (4b) of $\sigma$ together with the fact that $n \times [\nabla\phi]$, which is essentially the tangential part of $[\nabla\phi]$, can be readily calculated as $n \times \nabla\mu$. Equation (20) is usually referred to as the Helmholtz relation. Using the transformation rules (19), Eq. (18) becomes

$$\begin{aligned}
\partial_k'\phi^Q &= \iint_Q \{\psi[(n_j\partial_k - n_k\partial_j)\tilde{w}_j - (1-M^2)\Omega^2 n_k\mu] \\
&- w_k(n_j\tilde{\partial}_j\psi)\}\, dS_x - \int_{\partial Q} dx_i \epsilon_{ijk} [\psi\tilde{w}_j - \mu\tilde{\partial}_j\psi]
\end{aligned} \quad (21)$$

Since the operator $(n_j\partial_k - n_k\partial_j)$ (like the operator $n \times \nabla$) can be applied to a function defined only on the surface $Q$, this expression clearly illustrates how $(\partial_k' \phi)$ can be separated out into surface and line integrals with just the knowledge of the surface singularity distributions $\sigma$ and $\mu$. Indeed, comparing the factors multiplying the kernel function $\psi$ in Eq. (21) and (17), we see that the jump in the conormal derivative of $\partial_k \phi$, denoted $u_k$, may be expressed in the form

$$u_k = [n_j \tilde{\partial}_j \partial_k \phi] = (n_j \partial_k - n_k \partial_j)\tilde{w}_j - (1 - M^2)\Omega^2 n_k \mu \qquad (22)$$

In this way, Eq. (21) can be written

$$\partial_k' \phi^Q = \iint_Q [\psi u_k - w_k \tilde{n}_j \partial_j \psi] \, dS_x - \int_{\partial Q} dx_i \epsilon_{ijk} \psi \tilde{w}_j + \int_{\partial Q} dx_i \epsilon_{ijk} \mu (\tilde{\partial}_j \psi) \qquad (23)$$

This formula, with $u_k$ given by Eq. (22) and $w_k$ given by Eq. (20), is our final result. The two line integrals that appear are called the "edge-jet" and "line-vortex" terms, respectively. The line-vortex term is responsible for what has been called the hypersingular behavior of the original integral, Eq. (13).

The edge-jet term, $-\int_{\partial Q} dx_i \epsilon_{ijk} \psi \tilde{w}_j$ has the following interpretation. We will suppose that we are dealing with steady flow ($\omega = 0$) at zero Mach number ($M = 0$) and that the doublet strength $\mu = 0$. If the surface element $Q$ is flat and is assigned a constant source strength $\sigma$, the total rate of mass production will be $\sigma A_Q$, where $A_Q$ is the area of $Q$. For field points near the boundary of $Q$, all of the source elements ($\sigma$ d$S$) will combine in concert to induce a velocity component in the plane of $Q$ and perpendicular to its edge, having a magnitude proportional to log($g$), where $g$ is the distance from the field point to $\partial Q$ the boundary of $Q$. This can be seen setting $\mu = 0$ and $\sigma =$ const in Eq. (23), obtaining first $\boldsymbol{w} = \sigma \boldsymbol{n}$ from Eq. (20), then $\boldsymbol{u} = 0$ from Eq. (22), and finally the result

$$\partial_k' \phi^Q = -\iint_Q \sigma n_k (n_j \partial_j \psi) - \int_{\partial Q} dx_i \epsilon_{ijk} \psi \sigma n_j = -\sigma \iint_Q n_k \frac{\partial \psi}{\partial n} - \sigma \int_{\partial Q} (d\boldsymbol{x} \times \boldsymbol{n})_k \psi \qquad (24)$$

The line integral in this expression produces the log-singular term lying in the plane of $Q$ and perpendicular to the boundary $\partial Q$ for field points $\boldsymbol{x}'$ close to $\partial Q$. At field points near to $Q$ but far away from $\partial Q$, there is enough destructive interference of the velocity induced by the source elements to eliminate this singular behavior.

It is worth remarking that when total mass production for this source-only panel is computed by evaluating the far-field flux induced by each term, the term $-\iint_Q w_k \partial\psi/\partial\tilde{n} \, dS_x$ contributes $(1/3)\sigma A_Q$ to the total mass production, while $-\int_{\partial Q} dx_i \epsilon_{ijk} \psi \tilde{w}_j$ contributes $(2/3)\sigma A_Q$, the other two terms contributing nothing (see the Appendix for the details of the calculation). This observation argues compellingly against the dropping of the edge-jet term except when explicit matching of the edge velocity jump $w_i$ is performed. This situation is analogous to the case of the the second line integral term, $\int_{\partial Q} dx_i \epsilon_{ijk} \mu \tilde{\partial}_j \psi$, the line-vortex term. If this term is to be dropped, the doublet strength must be carefully matched at panel edges and along network abutments. If the line-vortex term is dropped without doing this, the resulting velocity field will no longer be irrotational. [This follows from the fact that the full expression (23) is irrotational being the gradient of Eq. (12), while the line-vortex term $\int_{\partial Q} \mu \, d\boldsymbol{x} \times \tilde{\nabla} \psi$ clearly has nonzero curl as can be determined from direct computation.] On the other hand, if the line-vortex term is retained, it will generate a vortical flow around the boundary line $\partial Q$ that becomes infinite as $1/g$ in the limit as the distance $g$ tends to zero.

## III. Linear Elasticity Analysis

The statement and proof of the integration by parts results for linear elasticity requires first the development of some notational machinery. Given a displacement field $u_i(\boldsymbol{x})$, the associated stress tensor field $\tau_{ij}(\boldsymbol{x})$ is given by the formula $t_{ijk} u_k$ where the stress operator $t_{ijk}$ is defined in terms of the Lamé coefficients $\lambda$ and $\mu$ by

$$t_{ijk} = \lambda \delta_{ij} \partial_k + \mu \delta_{ik} \partial_j + \mu \delta_{jk} \partial_i \qquad (25)$$

The differential equation satisfied by $u_k$ is the time-transformed Navier equation of linear elasticity:

$$\partial_j (t_{ijk} u_k) + \rho \omega^2 u_i = 0 \qquad (26)$$

where $\rho$ is the density and $\omega$ is the vibrational frequency. This can be recast as

$$L_{ik} u_k + \rho \omega^2 u_i = 0 \qquad (27)$$

where the operator $L_{ik}$ is given by

$$L_{ik} = \partial_l t_{ljk} = (\lambda + \mu) \partial_i \partial_k + \mu \delta_{ik} \partial_l \partial_l \qquad (28)$$

The Somiglian tensor (Green's tensor) $G_{kj}$ is defined as that tensor satisfying the relation

$$(L_{ik} + \rho \omega^2 \delta_{ik}) G_{kj} = \delta_{ij} \delta(\boldsymbol{x} - \boldsymbol{x}') \qquad (29)$$

It may be calculated directly using the methods of Ref. 5, Chap. IV, Sec. 28. The result of that calculation, readily verified by applying the operator $(L_{ik} + \rho\omega^2 \delta_{ik})$, is the formula for $G_{kj}$:

$$G_{kj} = \frac{1}{\rho \omega^2} \partial_k \partial_j (\psi_2 - \psi_1) + \frac{\delta_{jk}}{\mu} \psi_2 \qquad (30)$$

where the functions $\psi_\alpha$ are fundamental singularities for the Helmholtz equation at wave speeds $c_1 = [(\lambda + 2\mu)/\rho]^{1/2}$ and $c_2 = [\mu/\rho]^{1/2}$; that is, $(\nabla^2 + \omega^2/c_\alpha^2)\psi_\alpha = \delta(\boldsymbol{x} - \boldsymbol{x}')$. These are given explicitly by

$$\psi_\alpha = -\frac{\exp(-\iota \omega R / c_\alpha)}{4\pi R}, \qquad R = \|\boldsymbol{x} - \boldsymbol{x}'\| \qquad (31)$$

In the limit as $\omega \to 0$, $\psi_2 - \psi_1 = \iota\omega(c_2^{-1} - c_1^{-1})/4\pi + R\omega^2 (c_2^{-2} - c_1^{-2})/8\pi + \mathcal{O}(\omega^3)$. Consequently, at $\omega = 0$ we find after some manipulation,

$$G_{kj}\bigg|_{\omega=0} = \frac{c_2^{-2} - c_1^{-2}}{8\pi\rho} \partial_k \partial_j R - \frac{\delta_{jk}}{4\pi\mu} \frac{1}{R} = \frac{1}{4\pi\mu} \left[ \frac{\lambda + \mu}{2(\lambda + 2\mu)} \partial_k \partial_j R - \frac{\delta_{jk}}{4\pi\mu} \right]$$

which agrees essentially with Eq. (5.8a), p. 222, Ref. 4. The integral representation result analogous to Eq. (11), the Betti identity, states that any displacement field $u_k$ satisfying Eq. (27) in a region $B$ with boundary $S$ may be expressed in terms of the boundary values of $u_k$ and $n_j t_{ijk} u_k$ (the surface traction) by

$$u_l(\boldsymbol{x}') = \iint_S n_j(\boldsymbol{x}) [(t_{ijk} u_k) G_{il}(\boldsymbol{x} - \boldsymbol{x}') - u_i(\boldsymbol{x}) t_{ijk} G_{kl}(\boldsymbol{x} - \boldsymbol{x}')] \, dS_x \qquad (32)$$

Allowing for slight differences in notation, an essentially similar formula may be found in Eq. (5.17), p. 222, Ref. 4. In Eq. (32), $n_j(\boldsymbol{x})$ is the *interior* normal at the point $\boldsymbol{x}$ on $S$, pointing into the region $B$ in which $u_l$ satisfies Eq. (27) (see Fig. 2). Proceeding as before, we consider the contribution to

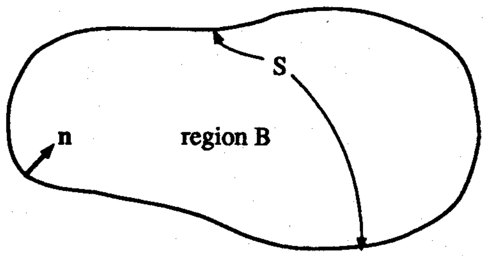



**Fig. 2 Concepts of the displacement representation formula (8).**

the preceding integral made by a surface element $Q$ of $S$. Denoting this by $u_i^Q$, we compute the gradient $u_{i,r}^Q$ as follows:

$$u_{i,r}^Q = \partial_r' u_i^Q = -\int\int_Q n_j[(t_{ijk}u_k)(\partial_r G_{il}) - u_i(\partial_r t_{ijk} G_{kl})]\,\mathrm{d}S_x \quad (33)$$

where we have used the antisymmetry rule $\partial_r' G_{il} = -\partial_r G_{il}$, since $G_{il}$ is a function of $(\boldsymbol{x} - \boldsymbol{x}')$. Proceeding as before, we apply Stokes' theorem in the form

$$\int_{\partial Q} \mathrm{d}x_\alpha\, \epsilon_{\alpha jr} f = \int\int_Q (n_j\partial_r - n_r\partial_j) f\,\mathrm{d}S_x$$

Taking $f$ to be the expression in the square brackets in Eq. (32), we obtain,

$$\begin{aligned}\int_{\partial Q} &\mathrm{d}x_\alpha \epsilon_{\alpha jr}[(t_{ijk}u_k)G_{il} - u_i(t_{ijk}G_{kl})]\\
&= \int\int_Q (n_j\partial_r - n_r\partial_j)G_{il}(t_{ijk}u_k)\,\mathrm{d}S_x\\
&+ \int\int_Q G_{il}[(n_j\partial_r - n_r\partial_j)t_{ijk}u_k)\,\mathrm{d}S_x\\
&- \int\int_Q (n_j\partial_r - n_r\partial_j)u_i(t_{ijk}G_{kl})\,\mathrm{d}S_x\\
&- \int\int_Q u_i[(n_j\partial_r - n_r\partial_j)t_{ijk}G_{kl}]\,\mathrm{d}S_x \qquad (34)\end{aligned}$$

As before, the second term and the sixth term on the right cancel, here by virtue of the general reciprocity relation $(\partial_j v_i)(t_{ijk}u_k) = (\partial_j u_i)(t_{ijk}v_k)$ applied to $v_k = G_{kl}$. Rearranging what remains, we get

$$\begin{aligned}\int_{\partial Q} &\mathrm{d}x_\alpha \epsilon_{\alpha jr}[(t_{ijk}u_k)G_{il} - u_i(t_{ijk}G_{kl})]\\
&= \int\int_Q n_j[(\partial_r G_{il})(t_{ijk}u_k) - u_i(\partial_r t_{ijk}G_{kl})]\,\mathrm{d}S_x\\
&+ \int\int_Q n_r[u_i(\partial_j t_{ijk}G_{kl}) - G_{il}(\partial_j t_{ijk}u_k)]\,\mathrm{d}S_x\\
&+ \int\int_Q n_j[G_{il}(\partial_r t_{ijk}u_k) - (\partial_r u_i)(t_{ijk}G_{kl})]\,\mathrm{d}S_x \qquad (35)\end{aligned}$$

Notice that the first line on the right exactly matches $(-u_{i,r}^Q)$ as given by Eq. (33). Solving for $u_{i,r}^Q$, we rearrange the terms of this equation to obtain a result analogous to Eq. (18), using the result $\partial_j t_{ijk}G_{kl} = -\rho\omega^2 G_{il}$ along the way:

$$\begin{aligned}u_{i,r}^Q = &\int\int_Q \{G_{il}[(n_j\partial_r - n_r\partial_j)(t_{ijk}u_k) - n_r\rho\omega^2 u_i]\\
&- (\partial_r u_i)(n_j t_{ijk}G_{kl})\}\,\mathrm{d}S_x\\
&- \int_{\partial Q} \mathrm{d}x_\alpha\, \epsilon_{\alpha jr}\,[(t_{ijk}u_k)G_{il} - u_i(t_{ijk}G_{kl})] \qquad (36)\end{aligned}$$

This is a useful representation of $u_{i,r}^Q$ because it permits its calculation solely from the knowledge of the surface traction $T_i = n_j t_{ijk}u_k$ and the values of $u_k$ on $Q$. To see why this is so, note first that $u_{k,i}$ can be written as the sum of two terms, involving tangential and normal derivatives of $u_k$:

$$u_{k,i} = (\delta_{ij} - n_i n_j)u_{k,j} + n_i(n_j\partial_j u_k) = u_{k,i}^{(t)} + n_i(n_j\partial_j u_k) \quad (37)$$

and that the first of these terms, $u_{k,i}^{(t)}$ can be computed solely from the knowledge of $u_k$ on $Q$. We will now show how to compute the other term from $T_i$. Writing $T_i$ as the sum of the two terms,

$$T_i \overset{\mathrm{def}}{=} \lambda n_i u_{k,k} + \mu n_k u_{k,i} + \mu n_k u_{i,k} = T_i^{(t)} + T_i^{(n)} \quad (38)$$

where we define the separate parts by

$$\begin{aligned}T_i^{(t)} &= \lambda n_i[\partial_k u_k - n_j(\partial_j u_k)n_k] + \mu(\delta_{ij} - n_i n_j)(\partial_j u_k)n_k\\
&= \lambda n_i[u_{k,k}^{(t)}] + \mu u_{k,i}^{(t)} n_k \qquad (39)\end{aligned}$$

$$T_i^{(n)} = \mu n_k \partial_k u_i + (\lambda + \mu)n_i[n_j(\partial_j u_k)n_k] \quad (40)$$

we see immediately that $T_i^{(t)}$ is computable from $u_{k,i}^{(t)}$. Thus, given $T_i$ and $u_k$ on $Q$, we can compute $T_i^{(n)}$ from $T_i^{(n)} = T_i - T_i^{(t)}$. Noting that as a consequence of Eq. (40), $n_i T_i^{(n)} = (\lambda + 2\mu)n_j(\partial_j u_k)n_k$, we find that Eq. (40) can be solved for $n_k\partial_k u_i$ in terms of $T_i^{(n)}$:

$$n_k\partial_k u_i = \frac{1}{\mu}\left\{T_i^{(n)} - \frac{\lambda + \mu}{\lambda + 2\mu}\, n_i(n_j T_j^{(t)})\right\} \quad (41)$$

With the appropriate substitution of indices in Eq. (37), this completes the description of how $u_{k,i}$ can be computed from $u_k$ and $T_i$ on $Q$. The sequence of formulas (37–41) comprise the elastic analog of the Helmholtz relation [Eq. (20)], of potential flow. Of course this also means that the stress tensor $\tau_{ij} = t_{ijk}u_k$ can be computed from the surface values of $u_k$ and $T_i$ as well. As we shall see, this fact is of some utility in the elimination of the line-integral terms.

Using the differential equation (27) satisfied by $u_i$, Eq. (36) can also be written as

$$\begin{aligned}u_{i,r}^Q = &\int\int_Q n_j[G_{il}(t_{ijk}u_{k,r}) - u_{i,r}(t_{ijk}G_{kl})]\,\mathrm{d}S_x\\
&- \int_{\partial Q} \mathrm{d}x_\alpha \epsilon_{\alpha jr}[(t_{ijk}u_k)G_{il} - u_i(t_{ijk}G_{kl})] \qquad (42)\end{aligned}$$

The terms on the first line are exactly what we would get by applying the Betti identity to $v_i = u_{i,r}$, a function that also satisfies the Navier equations (27). However, in the actual application of these results in a collocation code, the line integrals may not be discarded without first ensuring that appropriate matching conditions are satisfied. Analogous to the line-vortex term, the hypersingular "edge dislocation" term $\int_{\partial Q} \mathrm{d}x_\alpha \epsilon_{\alpha jr} u_i(t_{ijk}G_{kl})$ may be discarded only if the finite dimensional representation of the surface displacement satisfies matching (continuity) conditions along panel and network boundaries. If these conditions are not explicitly imposed, and the dislocation term is discarded, the strain fields $\epsilon_{kl}$ computed from $u_{l,r}$ will fail to satisfy compatibility. Analogous to the edge-jet term, the "stress jump" term $-\int_{\partial Q} \mathrm{d}x_\alpha \epsilon_{\alpha jr}(t_{ijk}u_k)G_{il}$ may be discarded only if the representation of the stress tensor $\tau_{ij} = t_{ijk}u_k$ in terms of surface displacements and surface tractions [see Eqs. (37–41)] satisfies matching conditions along panel and network boundaries.

## IV. Imposition of Matching Conditions

In the previous two sections, we have seen how the hypersingular integrals of the velocity or displacement gradient representations can be decomposed into surface and line integrals,

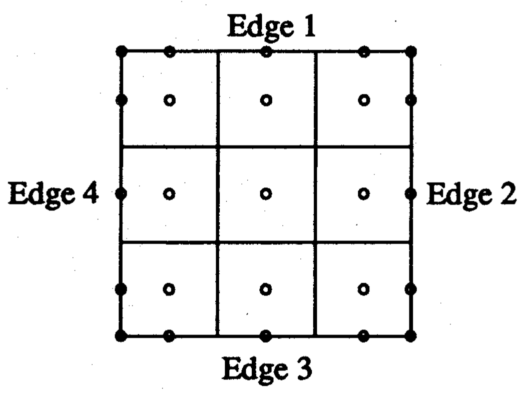



**Fig. 3 Schematic for a nine-panel doublet network, o = doublet parameter locations.**

with the line integrals exhibiting all of the singular behavior [both the hypersingular $1/g$ behavior and the milder $\log(g)$ singularity], the surface integrals being no more singular than the basic representation formulas for potential or displacement. Now provided that appropriate matching conditions are satisfied, the line integrals contributed by two adjacent panels will combine with opposite sign and cancel. In the case of potential flow, the matching conditions are imposed along any edge at which the finite dimensional representation of doublet strength $\mu$ or velocity jump $w$ might be discontinuous.

Line vortex removal:

$$\sum_k s_k \mu \,|_{Q_k} = 0 \tag{43a}$$

Edge jet removal:

$$\sum_k s_k w \,|_{Q_k} = 0 \tag{43b}$$

Here, the sum extends over all surface elements (panels) that meet along the given edge, the numbers $s_k = \pm 1$ reflect the relative orientation of those surface elements. On any given surface element $Q_k$, $w|_{Q_k}$ is computed from $\sigma$ and $\mu$ using Eq. (20). It should be clear that the removal of the edge-jet term, even in the interior of a network, will require a continuous surface normal, a continuous source representation, and a $C^1$ doublet representation. Because of the technical difficulty in achieving these objectives, edge-jet removal is not frequently done, although it is clear that the imposition of such continuity requirements is critical to the improvement in order of panel methods. Moreover, because the velocity induced by an edge jet exhibits a logarithmic singularity as the edge jet is approached, it is common practice in aerodynamic panel codes to recede any edge control points (collocation points) at least 10% of a panel diameter away from the edges.

The matching conditions for the Navier elasticity equations take the following forms.

Edge dislocation removal:

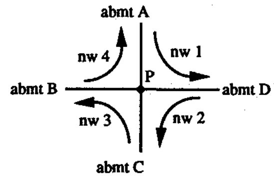
$$\sum_k s_k \boldsymbol{u} \,|_{Q_k} = 0 \tag{44a}$$

Stress jump removal:

$$\sum_k s_k \tau \,|_{Q_k} = 0 \tag{44b}$$

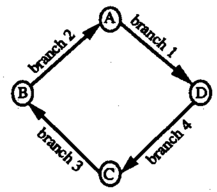
As in the case of potential flow, the removal of the stress jump term (which is analogous to the edge-jet term) requires a continuous normal, a continuous representation for surface traction $T_i$ and a $C^1$ representation for the surface displacement $u_i$. As in the case of the edge-jet term, it is clear that when the stress jump term is not removed, edge-collocation points should be moved in slightly from the panel edges.

Now the imposition of the matching conditions at an edge in the interior of a network is a fairly straightforward process.




| Abutment | Matching Condition at P |
|---|---|
| A | $\mu_1 - \mu_2 = 0$ |
| B | $\mu_2 - \mu_3 = 0$ |
| C | $\mu_3 - \mu_4 = 0$ |
| D | $\mu_4 - \mu_1 = 0$ |

**Fig. 4 An abutment intersection: four networks meeting at a point.**




| Node | Kirchhoff Nodal Constraints on Branch Currents |
|---|---|
| A | $I_1 - I_2 = 0$ |
| B | $I_2 - I_3 = 0$ |
| C | $I_3 - I_4 = 0$ |
| D | $I_4 - I_1 = 0$ |

**Fig. 5 Directed graph and matching condition assignments for the abutment intersection of Fig. 4.**

All that is required is that sufficiently continuous finite dimensional representations be found for $\mu$ and $\sigma$, $\boldsymbol{u}$ and $\boldsymbol{T}$. Even along network abutments (lines along which two or more networks meet), line-vortex and edge-dislocation removal is fairly straightforward. What is required is that the functional behavior of $\mu(\boldsymbol{u})$ along a network edge be completely determined by the values of the doublet (or displacement) parameters along that edge. For example, in the PAN-AIR code, the doublet strength along edge 1 of the schematic network in Fig. 3 is a piecewise quadratic function completely determined by the values of the five doublet parameters located on that edge. The dependence of $\mu|_{E_1}$ on these five parameters is specified by the fitting process called *edge spline construction.* When edge splines along a given edge are always constructed according to the same mathematical rules (independent of which network they belong to), the jump conditions (43) and (44) can be imposed all along the edge simply by requiring that they hold at the doublet parameter locations. Matching conditions of this sort can be imposed quite readily by entering the required matching condition constraints into the influence coefficient matrix explicitly, allowing (as is usually necessary) that the matching condition override any user provided collocation condition associated with the parameter location.

The procedure just outlined works very well along the interior of the network edges. At the ends of the network edges, things get a bit more complicated. Such a point, at which several abutments come together, is called an *abutment intersection.* The difficulty with abutment intersections is best illustrated by a simple case, the situation in which four networks come together at a point, as shown in Fig. 4. A quick examination of the four matching conditions reveals a redundancy: any three conditions imply the fourth. Thus, if we were to enter all four conditions into our influence coefficient matrix, we would obtain a singular matrix and the solution process would fail. In this case, the remedy is clear: we simply cast out one of the matching conditions, retaining only three. However, in the more general case of a complex abutment intersection, the situation is less clear. In those situations, a more comprehensive theory is clearly required.

In the more general situation we proceed as follows. Given an abutment intersection, we define a directed graph by constructing a small sphere around the abutment intersection and determining the intersection of the paneled configuration with the sphere. Abutments will intersect the sphere at points; these will be the nodes of our graph. Networks will intersect the sphere along lines; these will be the branches that connect the nodes of the graph. Furthermore, since each network is an oriented surface, the positive traversal of a network boundary provides a direction for each branch of the graph. As applied to the simple abutment intersection of Fig. 4, the process yields the graph and matching condition assignments of Fig. 5.

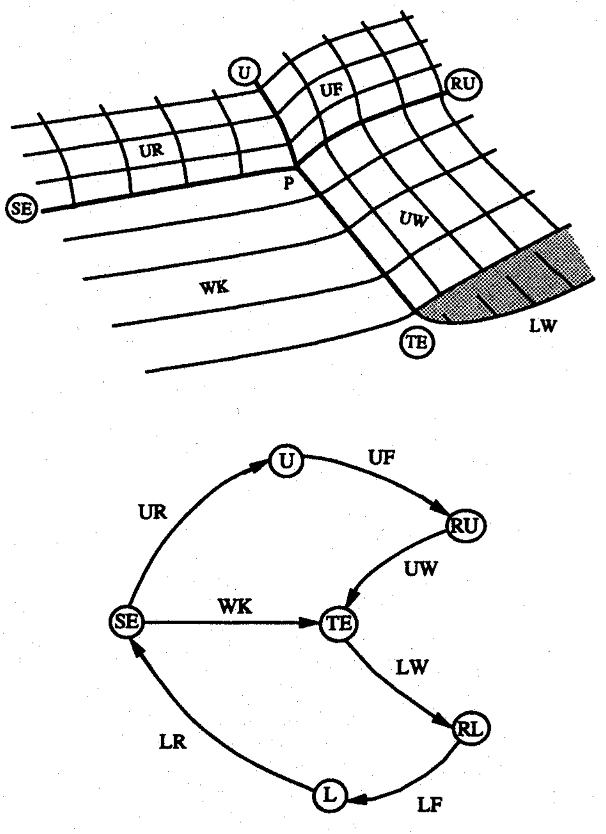



**Fig. 6 A more complex abutment intersection: trailing edge of a wing at the wing root.**

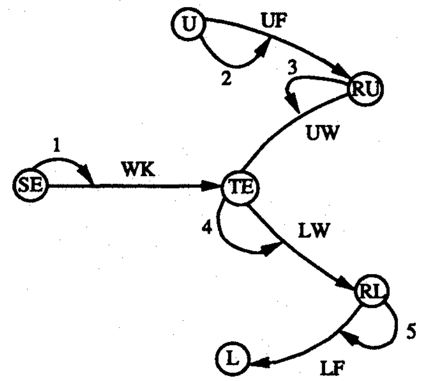



**Fig. 7 Directed graph and matching condition assignments for the abutment intersection of Fig. 6.**

As illustrated to the side of this figure, the doublet matching conditions correspond precisely to the Kirchhoff current laws for the nodes of the graph. Now it is a general theorem (cf. Ref. 6, Theorem 9-6 and Sec. 13-2) of network theory that for any connected graph with $M$ nodes, there are exactly $(M - 1)$ linearly independent Kirchhoff current balances. Consequently, any intersection of $M$ abutments that is represented by a connected graph in the fashion just outlined has associated with it $(M - 1)$ linearly independent matching conditions.

To illustrate the procedure further, consider the abutment intersection at the trailing edge of a wing where the wing is attached to the fuselage. A standard paneling of such a configuration would have seven networks coming together at a point $P$ as illustrated in Fig. 6. The branches (networks) and nodes (abutments) have been labeled as follows:

Networks/Branches
UF = Upper Forward Fuselage
UR = Upper Rear Fuselage
UW = Upper Wing Surface
LF = Lower Forward Fuselage
LR = Lower Rear Fuselage
LW = Lower Wing Surface
WK = Wake Surface

Abutments/Nodes
U = Abutment of networks UF and UW
RU = Abutment at wing root, upper surface
TE = Abutment at trailing edge of wing
RL = Abutment at wing root, lower surface
L = Abutment of networks LW and LF
SE = Abutment along side edge of the body

Now, in general, it is desirable to have an orderly procedure for replacing user-provided boundary conditions with the program-generated matching conditions. In this particular instance, we have the added complication of a wake network that has no user-provided boundary condition even though it does have a control point at the abutment intersection point $P$. Clearly the (nonexistent) boundary condition for this network must be overridden by some matching condition.

Thus we are led to the following two step procedure. (For a complete discussion of the graph theoretic terminology, and for associated algorithms, the reader is referred to Deo.[6]):

1) Given the graph, construct a spanning tree for the graph being careful to include those branches whose control points must receive a matching condition.

2) Defoliate that tree, one branch at a time by always selecting a node of degree 1 and removing that node and its associated branch. Perform this process using a prioritization scheme that always prefers the removal (and assignment of a corresponding matching condition) of a branch (such as a wake) that must receive a matching condition. The removal of each node and branch pair generates an association of network (control point) and corresponding matching-condition override.

To illustrate this algorithm, we apply it to the wing trailing-edge abutment intersection discussed above. A spanning tree and corresponding defoliation process are illustrated by Fig. 7. Notice that the wake is selected first for removal from the graph, since it must receive a matching condition. All of the other choices are somewhat arbitrary. Notice that the fourth branch removed (LW) can be removed only after three other branches (WK, UF, and UW) have been removed. One matching condition ($\mu_{LR} = \mu_{LF}$) is not explicitly imposed, being implied by the other five matching conditions.

Although the procedures outlined in this paper provide a good insight into the issues that affect doublet matching, they do not completely address the issue. For example, the treatment of abutment intersections that lie on a plane of configuration symmetry is rather complex, the treatment being different depending upon whether the solution is symmetric or antisymmetric with respect to the plane of symmetry. These issues are discussed more fully in Appendix H of the PAN AIR Theory Document, Version 3.[7]

## V. Reflections

The discussions of Secs. II and III present the derivation of the integration-by-parts formulas in a concise fashion, but do little to motivate those calculations. Having given the issue considerable thought, the following key ideas emerge as a guide to those seeking to perform similar derivations in different contexts.

First, we remark that in deriving such formulas, a form such as Eq. (11) involving the field function $\phi$ explicitly is always to be preferred over a representation such as Eq. (3) in which surface singularity functions have been defined. The main reason for this is that the form Eq. (3) conceals the appearance of the surface normal within the definition of $\sigma$.

Second, the form of Stokes' theorem that seems to be most useful is a slight rearrangement of the formula (14), given by

$$\iint_Q n_j \partial_k f \, dS = \iint_Q n_k \partial_j f \, dS + \int_{\partial Q} dx_i \, \epsilon_{ijk} f \tag{45}$$

Setting $f = g \cdot h$, in the classic spirit of integration by parts, we obtain after some rearrangement the result:

$$\iint_Q h n_j \partial_k g \, dS = \iint_Q h n_k \partial_j g \, dS + \iint_Q g(n_k \partial_j - n_j \partial_k) h \, dS + \int_{\partial Q} dx_i \, \epsilon_{ijk} g h \tag{46}$$

If I had understood the importance of this form of Stokes' theorem 14 years ago when I first looked into these issues, I would have saved much time and energy. To illustrate the use of this formula, we simply note that the left side is the doublet contribution to Eq. (13) provided we set $h = \phi$, $g = \tilde{\partial}_j \psi$. We obtain at once,

$$\iint_Q \phi(n_j \partial_k \tilde{\partial}_j \psi) \, dS_x = \iint_Q \phi n_k (\partial_j \tilde{\partial}_j \psi) \, dS_x + \iint_Q (\tilde{\partial}_j \psi)(n_k \partial_j - n_j \partial_k)\phi \, dS_x + \int_{\partial Q} dx_i \, \epsilon_{ijk} \phi \, \tilde{\partial}_j \psi \tag{47}$$

Using the differential equation (10) satisfied by $\psi$, we obtain after some change of notation,

$$\iint_Q \phi(n_j \partial_k \tilde{\partial}_j \psi) \, dS_x = -(1 - M^2)\Omega^2 \iint_Q n_k \phi \psi \, dS_x - \iint_Q [(n \times \nabla \phi) \times \tilde{\nabla} \psi]_k \, dS_x + \int_{\partial Q} \phi (dx \times \tilde{\nabla} \psi)_k \tag{48}$$

The line integral appearing on the right is just the line-vortex term, the other terms giving the regular part of the doublet's contribution to the velocity. The main effect of using formula (46) is to trade the operator $n_j \partial_k$ for $n_k \partial_j$, operating on $\tilde{\partial}_j \psi$. Then, because $\psi$ satisfies the differential equation (8), the right-hand side simplifies to a sum of less singular surface integrals and a discardable line integral.

## Appendix: Far-Field Mass Production From Eq. (24)

Assuming as in the discussion leading up to Eq. (24) that $\omega = 0$ and $M = 0$, we note that $\tilde{n}_j = n_j$ and that the mass produced by the two terms of Eq. (24) can be computed as $M = M_1 + M_2$ where

$$M_1 = \lim_{\Delta \to \infty} \left\{ \iint_{\|x'\| = \Delta} \left[ -\iint_Q \sigma n_k \frac{\partial \psi}{\partial n} \, dS_x \right] \frac{x_k'}{\|x'\|} \, dS_{x'} \right\}$$

$$M_2 = \lim_{\Delta \to \infty} \left\{ \iint_{\|x'\| = \Delta} \left[ -\int_{\partial Q} dx_i \epsilon_{ijk} \psi \sigma n_j \right] \frac{x_k'}{\|x'\|} \, dS_{x'} \right\}$$

Here, the integration is carried out over a large sphere $S_\Delta$ of radius $\Delta$ centered at the origin. Assuming for convenience that the panel $Q$ is circular, lies in the $x$-$y$ plane (so that $n_k = \delta_{k3}$) and is centered at the origin, the expression for $M_1$ simplifies to

$$M_1 = \lim_{\Delta \to \infty} \left\{ \frac{\sigma}{4\pi} \iint_{\|x'\| = \Delta} \left[ \iint_Q \frac{x_3' - x_3}{\|x' - x\|^3} \, dS_x \right] \frac{x_3'}{\|x'\|} \, dS_{x'} \right\}$$

The integrand in the inner integral is approximated via a Taylor series in $x$ by

$$\frac{x_3' - x_3}{\|x' - x\|^3} = \frac{x_3'}{\|x'\|^3} + \mathcal{O}(\|x'\|^{-3})$$

Using this, along with the result that on $\|x'\| = \Delta$, $dS_x = \Delta^2 \, d\Omega$, where $d\Omega$ is the element of solid angle, we find that $M_1$ contributes one-third of the total mass production:

$$\begin{aligned} M_1 &= \lim_{\Delta \to \infty} \left\{ \frac{\sigma}{4\pi} \iint_{\|x'\| = \Delta} \left[ \iint_Q \left( \frac{x_3'}{\|x'\|^3} + \mathcal{O}(\Delta^{-3}) \right) dS_x \right] \frac{x_3'}{\|x'\|} \Delta^2 \, d\Omega \right\} \\ &= \lim_{\Delta \to \infty} \left\{ \frac{\sigma}{4\pi} \iint_{\|x'\| = \Delta} \left[ \left( \frac{x_3'}{\|x'\|} \frac{x_3'}{\|x'\|} A_Q + \mathcal{O}(\Delta^{-1}) \right) \right] d\Omega \right\} \\ &= \frac{\sigma A_Q}{4\pi} \iint_{\|z\| = 1} z_3 z_3 \, d\Omega = \frac{\sigma A_Q}{4\pi} \frac{4\pi}{3} = \frac{\sigma A_Q}{3} \end{aligned}$$

Turning now to $M_2$, we write

$$M_2 = \frac{\sigma}{4\pi} \lim_{\Delta \to \infty} \left\{ \iint_{\|x'\| = \Delta} \left[ \int_{\partial Q} dx_i \epsilon_{ijk} \frac{1}{\|x' - x\|} \delta_{j3} \right] \frac{x_k'}{\|x'\|} \, dS_{x'} \right\}$$

Expanding $1/\|x' - x\|$ in a Taylor series, we have

$$\frac{1}{\|x' - x\|} = \frac{1}{\|x'\|} + \frac{x \cdot x'}{\|x'\|^3} + \mathcal{O}\left( \frac{1}{\|x'\|^3} \right)$$

Using this expansion, we see immediately that the $\mathcal{O}(\|x'\|^{-3})$ remainder contributes nothing to $M_2$ in the limit as $\Delta \to \infty$. Thus, we obtain

$$M_2 = \frac{\sigma}{4\pi} \lim_{\Delta \to \infty} \left\{ \iint_{\|x'\| = \Delta} \left[ \int_{\partial Q} dx_i \epsilon_{ijk} \delta_{j3} \left( \frac{1}{\|x'\|} + \frac{x \cdot x'}{\|x'\|^3} \right) \right] \frac{x_k'}{\|x'\|} \, dS_{x'} \right\}$$

Now $\int_{\partial Q} dx_i = 0$ whereas $\int_{\partial Q} dx_i x = \epsilon_{ipq} n_p e_q A_Q$, where $n_p = \delta_{p3}$. as in the preceding. Thus, we conclude that $M_2$ contributes two-thirds of the total mass production:

$$\begin{aligned} M_2 &= \frac{\sigma}{4\pi} \lim_{\Delta \to \infty} \left\{ \iint_{\|x'\| = \Delta} \left[ \epsilon_{ijk} \delta_{j3} \frac{x'}{\|x'\|^3} \cdot (\epsilon_{ipq} \delta_{p3} e_q A_Q) \right] \frac{x_k'}{\|x'\|} \, dS_{x'} \right\} \\ &= \frac{\sigma}{4\pi} \lim_{\Delta \to \infty} \left\{ \iint_{\|x'\| = \Delta} \frac{x_q'}{\|x'\|^3} \frac{x_k'}{\|x'\|} A_Q [\delta_{kq} - \delta_{k3} \delta_{q3}] \, dS_{x'} \right\} \\ &= \frac{\sigma A_Q}{4\pi} \iint_{\|z\| = 1} (1 - z_3 z_3) \, d\Omega = \frac{\sigma A_Q}{4\pi} \left( \frac{2}{3} 4\pi \right) = \frac{2\sigma A_Q}{3} \end{aligned}$$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